REALIZATION OF A SUM OF SEQUENCES BY A SUM GRAPH

BY

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ABSTRACT

It is shown that the realizability of the sequences $\phi=(a_1,\dots,a_n)$, $\psi=(b_1,\dots,b_n)$ and $\phi+\psi$ is a sufficient condition for the realizability of $\phi+\psi$ by a graph with a ϕ-factor if $b_i \leqq 1$ for $i=1,\dots,n$. The condition is not sufficient in general. A necessary and sufficient condition for the realizability of $\phi+\psi$ by a graph with a ϕ-factor is given for the case that ϕ is realizable by a star and isolated vertices.

1. Introduction

All graphs in this paper are finite, undirected and without loops or multiple edges.

$G = G(P_1,\cdots,P_n)$ denotes a labelled graph with n vertices $P_1,\cdots,P_n$. The degree $d_G(P_i)$, or $d(P_i)$ of a vertex P_i of a graph G is the number of edges of G, incident with P_i.

$\pi(G) = (d(P_1),\cdots,d(P_n))$ is the *degree sequence* of G. The finite sequence $\phi = (a_1,\cdots,a_n)$ is *realizable* if there exists a graph $G = G(P_1,\cdots,P_n)$ such that $\phi = \pi(G)$; G is then called a realization of ϕ.

$E(G)$ denotes the set of edges of G.

A graph $H(P_1,\cdots,P_n)$ is called a ϕ-factor of $G(P_1,\cdots,P_n)$ if $E(H) \subseteq E(G)$ and $\pi(H) = \phi$. If $\phi = (1,\cdots,1)$, then a ϕ-factor is called a 1-factor.

DEFINITION. Let $\phi(a_1,\cdots,a_n)$, $\psi=(b_1,\cdots,b_n)$ be two finite sequences; a realization $G(P_1,\cdots,P_n)$ of the sequence $\phi+\psi = (a_1+b_1,\cdots,a_n+b_n)$ is a (ϕ,ψ) *realization* of $(\phi+\psi)$ iff G has a ϕ-factor $H(P_1,\cdots,P_n)$.

Received November 28, 1972 and in revised form April 8, 1973

REMARK. If G is a (ϕ, ψ) realization then G has also a ψ-factor $K(P_1, \cdots, P_n)$; $E(K) = E(G) - E(H)$. Hence G is also called a *sum graph*.

A forest, as usual, is a graph without circuits, and a tree is a connected forest. A star is a tree with at most one vertex of degree greater than 1.

DEFINITION. A sequence $\phi = (a_1, \cdots, a_n)$ is *staric* if ϕ is realizable by a forest consisting of a star and isolated vertices, i.e., if ϕ is, up to a permutation, of the form $(m, \overbrace{1, \cdots, 1}^{m}, 0, \cdots, 0)$, where $0 \leqq m \leqq n-1$.

In this paper we shall investigate the following conjecture: If $\phi = (a_1, \cdots, a_n)$, $\psi = (b_1, \cdots, b_n)$ and $\phi + \psi$ are realizable sequences then $\phi + \psi$ has a (ϕ, ψ) realization. (The realizability of ϕ, ψ and $\phi + \psi$ is of course a necessary condition for the existence of a (ϕ, ψ) realization).

We shall show that the conjecture is true if $b_i \leqq 1$ for $i = 1, \cdots, n$ and also if $\phi + \psi$ is realizable by a forest, but that it is not true in general. We shall also provide a necessary and sufficient condition for the existence of a (ϕ, ψ) realization when ψ is staric.

2. (ϕ, ψ) realization I

Throughout this paper we shall use the following lemma:

LEMMA 1. *If $\phi = (a_1, \cdots, a_n)$ is a realizable sequence and $a_1 \geqq a_2 \geqq, \cdots, \geqq a_n$, then for every i, $1 \leqq i \leqq n$, there exists a realization $G_i(P_1, \cdots, P_n)$ of ϕ in which the set of vertices adjacent to P_i is*

$$\{P_k\colon 1 \leqq k \leqq a_i\}, \text{ if } a_i < i, \text{ or}$$

$$\{P_k\colon 1 \leqq k \leqq a_i + 1, k \neq i\} \text{ if } a_i \geqq i.$$

For proofs of Lemma 1 see [6, Lemma 6], [3, Lemma (3.2)].

THEOREM 1. *Let $\phi = (a_1, \cdots, a_n)$, $\psi = (b_1, \cdots, b_n)$ and $\phi + \psi$ be three realizable sequences and let $b_i \leqq 1$ $(i = 1, \cdots, n)$. Then $\phi + \psi$ has a (ϕ, ψ) realization.*

COROLLARY 1. *A realizable sequence $\phi_1 = (c_1, \cdots, c_n)$ has a realization G with a 1-factor iff $\phi = (c_1 - 1, \cdots, c_n - 1)$ is realizable.*

PROOF OF COROLLARY 1. Since ϕ_1 and ϕ are realizable, $\Sigma^n_{i=1} c_i$ and $\Sigma^n_{i=1}(c_i - 1)$ are even. Thus n is even, hence $\psi = (1, \cdots, 1)$ is realizable and we can apply Theorem 1 to $\phi_1 = \phi + \psi$.

REMARK. Corollary 1 was conjectured by Branko Grünbaum; see [4, p. 492].

PROOF OF THEOREM 1. Without loss of generality we may assume that both ϕ and $\phi+\psi$ are nonincreasing: we can obviously reorder the indices so that ϕ becomes nonincreasing. If ϕ is nonincreasing and for some i, $1 \leqq i < n$, $a_i = a_{i+1}$, $b_i = 0$ and $b_{i+1} = 1$, then we may interchange the indices i and $i+1$. (If $a_i > a_{i+1}$ then $a_i + b_i \geqq a_{i+1} + b_{i+1}$ since $b_{i+1} \leqq 1$.) By a finite number of such interchanges we obtain a nonincreasing reordering of both ϕ and $\phi+\psi$.

The proof will proceed by induction on n. If $n = 1$ then $a_1 = b_1 = 0$. Assume the theorem is true for all k, $k < n$.

Case a. For some i, $1 \leqq i \leqq n$, $b_i = 0$. Choose such an i. In order to simplify the notation, assume that $a_i < i$ and let $a_i = m$. (The case $a_i \geqq i$ is practically the same). ϕ is realizable, hence by Lemma 1, there exists a realization of ϕ in which P_i is adjacent to $P_1, P_2, \cdots, P_m$. Hence

$$\phi^* = (a_1 - 1, \cdots, a_m - 1, a_{m+1}, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$$

is realizable. $\phi + \psi$ is realizable, hence by Lemma 1 there exists a realization of $\phi+\psi$ where P_i is adjacent to $P_1, \cdots, P_m$. Let $\psi^* = (b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n)$. Then $\phi^* + \psi^*$

$$= (a_1 + b_1 - 1, \ldots, a_m + b_m - 1, a_{m+1} + b_{m+1}, \ldots, a_{i-1} + b_{i-1}, a_{i+1} + b_{i+1}, \ldots, a_n + b_n)$$

is realizable. Clearly ψ^* is realizable. By the induction hypothesis, $\phi^* + \psi^*$ has a (ϕ^*, ψ^*) realization $G^*(P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_n)$. By adding to G^* the vertex P_i and connecting it to $P_1, \cdots, P_m$ we obtain a (ϕ, ψ) realization $G(P_1, \cdots, P_n)$: if H^* is a ϕ^*-factor of G^*, then $H(P_1, \cdots, P_n)$ with $E(H) = E(H^*)) \cup \{(P_i, P_j): 1 \leqq j \leqq m\}$ is a ϕ-factor of G.

Case b. $b_1 = b_2 = \cdots = b_n = 1$.

In this case $a_1 + b_1 = a_1 + 1 \leqq n - 1$, hence $a_1 \leqq n - 2$. We also assume that $a_1 > 0$, since otherwise $\phi = 0$ and there is nothing to prove. Let $a_1 = m$.

ϕ is realizable; hence $\phi^* = (a_2 - 1, \cdots, a_{m+1} - 1, a_{m+2}, \cdots, a_n)$ is realizable (compare the proof of Case a).

Let $\psi^* = (b_2, \cdots, b_{m+1}, 0, b_{m+3}, \cdots, b_n)$. ψ^* is clearly realizable. $\phi + \psi$ is realizable, hence $\phi^* + \psi^*$ is realizable (compare the proof of Case a). By the induction hypothesis $\phi^* + \psi^*$ has a (ϕ^*, ψ^*) realization $G^*(P_2, \cdots, P_n)$. By adding to G^* the vertex P_1 and connecting it to $P_2, \cdots, P_{m+2}$ we obtain a (ϕ, ψ)-realization, since if $H^*(P_2, \cdots, P_n)$ is a ϕ^*-factor of G^*, then $H(P_1, \cdots, P_n)$ with $E(H) = E(H^*)$ $\cup \{(P_1, P_i): 2 \leqq i \leqq a_{m+1}\}$ is a ϕ-factor of G.

3. (ϕ, ψ) realization II

For Theorem 2 we need the following lemma.

LEMMA 2. *Let $\phi = (a_1, \cdots, a_n)$ be a sequence of nonnegative integers. If $1 \leqq k < n$, then ϕ is realizable by a forest of k trees iff $\sum_{i=1}^{n} a_i = 2(n-k)$ and there are fewer than k zeros in ϕ.*

The proof of the lemma is straightforward and is left to the reader.

THEOREM 2. *Let $\phi = (a_1, \cdots, a_n)$, $\psi = (b_1, \cdots, b_n)$ be two realizable sequences and suppose $\phi + \psi$ is realizable by a forest. Then $\phi + \psi$ has a (ϕ, ψ) realization.*

PROOF. Assume $\phi + \psi$ is realizable by a forest with k trees. If $a_i \leqq 1$ for $i = 1, \cdots, n$, or $b_i \leqq 1$ for $i = 1, \cdots, n$, then, by Theorem 1, $\phi + \psi$ has a (ϕ, ψ) realization. Assume therefore that $a_i > 1$ for some i, $b_j > 1$ for some j. This clearly implies that $k < n - 1$. The proof is by induction on n. If $n = 1$, then $a_1 = b_1 = 0$.

Assume the theorem is true for all m, $m < n$. A forest with at least one edge has at least two one-valent vertices. Therefore at least two terms in $\phi + \psi$ are equal to 1. Assume, without loss of generality, that $a_1 \geqq a_i$ for $i = 2, \cdots, n$, $a_1 > 1$, $a_n = 1$, $b_n = 0$. Define $\phi' = (a_1 - 1, a_2, \cdots, a_{n-1}) = (a'_1, \cdots, a'_{n-1})$. ϕ' is realizable, by Lemma 1. Define $\psi' = (b_1, b_2, \cdots, b_{n-1})$. ψ' is realizable.

$$\sum_{i=1}^{n-1} (a'_i + b_i) = \sum_{i=1}^{n} (a_i + b_i) - 2 = 2[(n-1) - k],$$

and there are fewer than k zeros in $\phi' + \psi'$. Therefore, $\phi' + \psi'$ is realizable by a forest, hence, by the induction hypothesis, $\phi' + \psi'$ has a (ϕ', ψ') realization $G'(P_1, P_2, \cdots, P_{n-1})$. Adding to G' the vertex P_n and connecting it to P_1 we obtain a (ϕ, ψ) realization. (Compare the proof of Theorem 1.)

REMARK. The (ϕ, ψ) realization need not be a forest, as can be seen from the following example: Let $\phi = (2,2,2,0,0)$ $\psi = (0,0,0,1,1)$. Then $\phi + \psi$ has a realization by a tree but the (unique) (ϕ, ψ) realization is not a forest.

4. A counterexample

Let $\phi = (3,2,2,1,0)$. ϕ has a unique realization G_1. Let $\psi = (1,0,0,2,1)$. ψ has a unique realization G_2. $\phi + \psi = (4,2,2,3,1)$. $\phi + \psi$ has a unique realization G. But (P_2, P_3) is an edge of G_1 and is not an edge of G, hence G is not a (ϕ, ψ) realization, i.e., $\phi + \psi$ has no (ϕ, ψ) realization.

5. A condition for (ϕ, ψ) realization

[3] P. Erdös and T. Gallai found the following necessary and sufficient

condition for a nonincreasing sequence $(a_1,\cdots,a_n)$ of nonnegative integers to be realizable:

$$\sum_{i=1}^{n} a_i \text{ is even;} \tag{1}$$

$$\sum_{i=1}^{j} a_i - j(j-1) \leqq \sum_{k=j+1}^{n} \min(j,a_k), \quad (j = 1,\cdots,n). \tag{2}$$

For English and French versions of this result of Erdös and Gallai, with various proofs, see [1, pp. 110–111], [2, pp. 427–429, 433–436, 499, ex. 6.47] and [5, pp. 59–61].

Condition (2) is equivalent to the following condition:

$$\sum_{i=1}^{j} a_i - j(j-1) \leqq (l-j)j + \sum_{k=l+1}^{n} a_k, \quad (j = 1,\cdots,n,\ l = j, j+1,\cdots,n) \tag{3}$$

[3, (1.4)].

For the case of an arbitrary sequence of nonnegative integers we have to replace (3) by

$$\sum_{i\in S} a_i \leqq \sum_{i\in T} a_i + s(n-1-t) \tag{4}$$

for all disjoint sets S, T such that $\phi \neq S \cup T \subseteq N$, where $N = \{1,\cdots,n\}$, s and t are the cardinalities of S and T respectively [6, condition (1.4)].

THEOREM 3. *Let $\phi = (a_1,\cdots,a_n)$ be a sequence of nonnegative integers. Let $\psi = (b_1,\cdots,b_n)$ be staric with $b_n = m > 1$, and assume $\phi + \psi$ is realizable. Define for every pair of disjoint sets $S, T \subset N$,*

$$d(S,T) = \begin{cases} \sum_{i\in S} b_i & \text{if } n \notin S \cup T \\ 0 & \text{if } n \in S \cup T. \end{cases}$$

Then $\phi + \psi$ has a (ϕ,ψ) realization iff

$$d(S,T) + \sum_{i\in S} a_i \leqq \sum_{i\in T} a_i + s(n-1-t) \tag{5}$$

for all disjoint sets $S, T \subset N$.

REMARK. Condition (5) implies condition (4) for ϕ, hence (5) is a sufficient condition for the realizability of ϕ.

PROOF. First we shall prove the necessity of condition (5).

If $n \in T \cup S$ then $d(S,T) = 0$, so in this case (5) is necessary for the realization of ϕ. Suppose $n \notin S \cup T$. Let G^* be a (ϕ, ψ) realization, with a ψ-factor K^*. The number of edges of K^* connecting P_n to vertices of $\{P_i : i \in T\}$ is evidently $\sum_{i \in T} b_i$. (Note that $b_i \leqq 1$ for $i \in T$.) By removing from G^* these edges of K^*, we obtain a graph G. In G we have, by (4):

$$\sum_{i \in S} d(P_i) \leqq \sum_{i \in T} d(P_i) + s(n-1-t),$$

but

$$\sum_{i \in S} d(P_i) = \sum_{i \in S} (a_i + b_i) = d(S,T) + \sum_{i \in S} a_i$$

and

$$\sum_{i \in T} d(P_i) = \sum_{i \in T} a_i,$$

hence

$$d(S,T) + \sum_{i \in S} a_i \leqq \sum_{i \in T} a_i + s(n-1-t).$$

Now we shall prove the sufficiency of condition (5): Let M be the subset of $\{1, \cdots, n-1\}$ such that that $i \in M$ iff $b_i = 1$. We assume that $M = \{i_1, \cdots, i_m\}$ where $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n-1$. Since $b_i \leqq 1$ for $i = 1, \cdots, n-1$, we may assume, without loss of generality, that $a_1 \geqq \cdots \geqq a_{n-1}$ and also $a_1 + b_1 \geqq \cdots \geqq a_{n-1} + b_{n-1}$. (Compare the proof of Theorem 1.)

Define, for $0 \leqq k \leqq m$, $\phi_k = (a_1^k, \cdots, a_n^k)$, where

$$a_i^k = \begin{cases} a_i + k & \text{if } i = n \\ a_i + 1 & \text{if } i = i_1, \cdots, i_k \\ a_i & \text{otherwise}. \end{cases}$$

Then $a_1^k \geqq a_2^k \geqq \cdots \geqq a_{n-1}^k$ $(k = 0, 1, \cdots, m)$.

In Lemma 3 we shall prove that ϕ_k is realizable for $0 \leqq k \leqq m$.

Define $i_0 = 0$, $i_{m+1} = n$. Then (as will be shown in Lemma 4) for some k, $0 \leqq k \leqq m$, we have $i_k \leqq a_n + k < i_{k+1}$. Choose such a k. By Lemma 1, ϕ_k has a realization G' in which P_n is adjacent to $P_1, P_2, \cdots, P_{a_n+k}$. Adding to G' the edges (P_{i_l}, P_n) for $l = k+1, \cdots, m$, we obtain a graph G, such that $\pi(G) = \phi + \psi$, and $\{(P_{i_l}, P_n) \mid 1 \leqq l \leqq m\} \subseteq E(G)$; i.e., G is a (ϕ, ψ) realization.

We will complete the proof by stating and proving Lemmas 3 and 4.

LEMMA 3. *Let $\phi = (a_1, \cdots, a_n)$, $\psi = (b_1, \cdots, b_n)$ be two sequences, satisfying all the requirements of Theorem 3, including condition (5). Define ϕ_k for $0 \leqq k \leqq m$ as in the proof of Theorem 3. Then ϕ_k is realizable for $0 \leqq k \leqq m$.*

PROOF. We must show that each sequence ϕ_k satisfies condition (4), i.e., for every two disjoint sets $S, T \subset N$

$$\sum_{i \in S} a_i^k \leqq \sum_{i \in T} a_i^k + s(n-1-t) \text{ (for } 0 \leqq k \leqq m) .$$

Case a. $n \in S$. Suppose that

$$\sum_S a_i^k > \sum_T a_i^k + s(n-1-t) . \tag{6}$$

Since we assume that $\phi + \psi$ is realizable, inequality (4) for $\phi_m = \phi + \psi$ yields

$$\sum_S a_i^m \leqq \sum_T a_i^m + s(n-1-t) . \tag{4'}$$

This is a contradiction, since

$$\sum_S a_i^m - \sum_S a_i^k \geqq a_n^m - a_n^k = m - k ,$$

and

$$\sum_T a_i^m + s(n-1-t) - \sum_T a_i^k - s(n-1-t) \leqq \sum_{i=1}^{n-1} a_i^m - \sum_{i=1}^{n-1} a_i^k = m-k .$$

Case b. $n \in T$. Suppose again that

$$\sum_S a_i^k > \sum_T a_i^k + s(n-1-t) .$$

Inequality (4) for $\phi_0 = \phi$ implies

$$\sum_S a_i \leqq \sum_T a_i + s(n-1-t) . \tag{4''}$$

This is again a contradiction, since

$$\sum_S a_i^k - \sum_S a_i \leqq \sum_{i=1}^{n-1} a_i^k - \sum_{i=1}^{n-1} a_i = k$$

and

$$\sum_T a_i^k - \sum_T a_i \geqq a_n^k - a_n = k .$$

Case c. $n \notin S \cup T$.

In this case inequality (5) yields

$$\sum_S a_i^k \leqq \sum_S a_i^m = d(S,T) + \sum_S a_i \leqq \sum_T a_i + s(n-1-t) \leqq \sum_T a_i^k + s(n-1-t).$$

LEMMA 4. *Let $i_1, \cdots, i_m$ be integers, $1 \leqq i_1 < \cdots < i_m \leqq n-1$, and let a_n be* *a number. If* $a_n \geqq i_1$, *and* $a_n + m < i_m$ *then for some* $k, 1 \leqq k < m$, *we have* $i_k \leqq a_n + k < i_{k+1}$.

PROOF. If $a_n + m < i_m$, then $a_n + (m-1) < i_m$. Let l be the smallest nonnegative index such that $a_n + l < i_{l+1}$. By assumption, $a_n + 0 \geqq i_1$ hence $0 < l \leqq m - 1$. Then $i_l \leqq a_n + l$, since otherwise $a_n + (l-1) < a_n + l < i_l$, a contradiction to the minimality of l.

ACKNOWLEDGEMENT

I wish to express my deep gratitude to Professor Micha A. Perles who helped me greatly throughout this research with helpful criticism.

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